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THE WEAK BRUHAT ORDER OF S_Σ, CONSISTENT SETS, AND CATALAN NUMBERS*

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Abstract. *Chains* in the *weak Bruhat order* β of S_Σ (the symmetric group on Σ) belong to the class of subsets of S_Σ over which unrestricted choice necessarily produces transitive relations under pairwise simple majority vote (*consistent sets*). If for $A \subset S_\Sigma$ we let $T(A) \equiv \cup_{p \in A} T(p)$ where $T(p) = \{(p_i, p_j, p_k) | i < j < k\}$ and $\Psi(A) \equiv \{w \in S_\Sigma | T(w) \subset T(A)\}$ the following theorem (among others) is obtained.

THEOREM. *For all* $q \in S_\Sigma$, *if* A *is a saturated chain under* β *then* $\Psi(qA)$ *is an upper semimodular sublattice of cardinality* $|\Psi(qA)| \leqq \frac{1}{|\Sigma|+1}\binom{2|\Sigma|}{|\Sigma|} \equiv$ *The* $|\Sigma|$*th Catalan number.*

From the Arrow's Impossibility Theorem point of view, the results obtained here indicate that majority rule produces transitive results if the collection of voters as a whole can be partitioned into no more than $(|\Sigma|^2 + |\Sigma|)/2$ groups which can be ordered according to the level of disagreement they have with respect to a fixed permutation p. On the other hand, by viewing S_Σ as a *Coxeter group* a "novel" combinatorial interpretation of the collection of maximal chains that can be obtained from one another by using only one type of Coxeter transformation is obtained.

Key words. weak Bruhat order, upper semimodular lattice, Catalan numbers, Arrow's Impossibility Theorem, Coxeter groups

AMS(MOS) subject classifications. 05, 06

Introduction. The Marquis de Condorcet recognized nearly 200 years ago [12] that majority rule can produce intransitive group preferences if the domain of possible (transitive) individual preference orderings is unrestricted. This phenomenon is commonly known as the voting paradox (see Black [9] and Riker [20] for an excellent historical account).

Domains for which the simple majority rule produces transitive results are called here "Transitive Simple Majority" domains (TSM). The study of the structure and cardinality of TSM domains has proven to be a combinatorial problem of an unusual sort (Abello [1], [2], [4], Abello and Johnson [3], Arrow [5], Black [9], Fishburn [15], Good [17], Ward [25]).

By restricting our attention to TSM domains that are subsets of the symmetric group (called here "consistent sets") we have given general constructions that produce "consistent" sets of greater cardinality than all those offered in the past (Abello [2], Abello and Johnson [3]). All the constructed sets are maximally transitive and they achieve the best known (uniform) general lower bound.

A unified view of several seemingly different constructions of "consistent" sets has been obtained by Abello [1] via the weak Bruhat order, β, of S_n (Bourbaki [10], Lehmann [19], Savage [21], Yanagimoto and Okamoto [26]).

In this paper we will present the only known global structural properties of "consistent" sets. Namely, we prove that each maximal "consistent" set that contains a maximal chain in β is an upper semimodular sublattice of $\langle S_n, \beta \rangle$. This offers a "novel"

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combinatorial interpretation of each collection of maximal chains in β whose elemen can be obtained from one another by using one type of *Coxeter transformation* (Benso and Grove [6], Coxeter and Moser [13]). Moreover, we prove that each of these maxim: transitive sets has cardinality bounded by the nth *Catalan number*. This provides th unique nontrivial upper bound known to date.

We must remark that even though we restrict our attention to subsets of the syn metric group, many of the ideas contained here are extendable to the more gener: domains discussed in Chapter 1 of Abello [4], as they stand or with modification.

1. Preliminaries. Let $\langle \Sigma, \leqq \rangle$ be a totally ordered set of symbols of cardinalit $|\Sigma| = n \in Z^+$ and S_Σ the group of permutations on Σ (we will be using one line notatio for permutations).

DEFINITION 1.1. A set $\{u, v, w\} \subset S_\Sigma$ is called a *cyclic* three-set if there are thre symbols $x, y, z \in \Sigma$ such that $u^{-1}(x) < u^{-1}(y) < u^{-1}(z)$, $v^{-1}(y) < v^{-1}(z) < v^{-1}(x$ $w^{-1}(z) < w^{-1}(x) < w^{-1}(y)$.

DEFINITION 1.2. A subset C of S_Σ is called *consistent* if it contains no cyclic three set; otherwise C is called a *cyclic* set.

DEFINITION 1.3.

i. For $p \in S_\Sigma$, let:

$$T(p) \equiv \{(x,y,z) \mid p^{-1}(x) < p^{-1}(y) < p^{-1}(z)\};$$

$$\Gamma(p) \equiv \{(x,y) \mid p^{-1}(x) < p^{-1}(y)\};$$

$$\tau(p) \equiv \{(x,y) \in \Gamma(p) \mid p^{-1}(x) + 1 = p^{-1}(y)\}.$$

We will refer to $T(p)$, $\Gamma(p)$, and $\tau(p)$ as the sets of triples, pairs, and admissible adjacer transpositions determined by p, respectively. If $t \in \tau(p)$ then $t(p)$ will denote the pe mutation obtained from p by interchanging the symbols x and y where $(x, y) = t$.

ii. For $C \subseteq S_\Sigma$, let $T(C) \equiv \bigcup_{p \in C} T(p)$, $\Gamma(C) \equiv \bigcup_{p \in C} \Gamma(p)$, $\tau(C) \equiv \bigcup_{p \in C} \tau(p)$. Not that $|T(p)| = \binom{|\Sigma|}{3}$ for $|\Sigma| \geqq 3$. We will say that $T(C)$ is a *cyclic* or *consistent* set c triples depending on whether C is a *cyclic* or *consistent* subset of S_Σ, respectively.

The following are some elementary properties of consistent sets.

FACT 1.1.

i. *Any subset of a consistent set is consistent and any superset of a cyclic set is cyclic*

ii. *The intersection of consistent sets is consistent but their union is not always cor sistent.*

iii. $|T(S_\Sigma)|$ = *the number of different* 3*-permutations out of a set of* $|\Sigma|$*-elements*

iv. *If* C *is a consistent subset of* S_Σ *then* $|T(C)| \leqq 4\binom{|\Sigma|}{3}$.

2. A closure operator on S_Σ. The results in this section are independent of con sistency.

DEFINITION 2.1.

i. Let $\Psi: 2^{S_\Sigma} \to 2^{S_\Sigma}$ be given by $\Psi(A) = M_A = \{w \in S_\Sigma \mid T(w) \subseteq T(A)\}$.

ii. If $A \subseteq S_\Sigma$ is such that $\Psi(A) = A$ then A is called a *closed* subset and if $K \subseteq$ satisfies that $\Psi(K) = \Psi(A)$ where $|K| = \min |B|$ (taken over all subsets B of A suc that $T(B) = T(A)$), then K is called a *kernel* for A.

Let $C_K = \{A \subseteq S_\Sigma \mid K \text{ is a kernel for } A\}$. The following facts are immediate fron the preceding definitions.

FACT 2.1.

i. Ψ *is a closure operator on* S_Σ, *namely*, $A \subseteq \Psi(A)$; *if* $A \subseteq B$ *then* $\Psi(A) \subseteq \Psi(B$ *and* $\Psi^2(A) = \Psi(A)$.

ii. *There is a unique closed set in* C_K, *namely*, $X_K = \Psi(K)$.

iii. *If* K *and* K′ *are kernels for* C_K *and* $C_{K'}$, *respectively, and* $T(K) \subset T(K')$ *then* $\Psi(K) \subset \Psi(K')$.

The preceding result means that a *closed* set is completely determined by its kernels; moreover, any kernel K of a *closed* set X_K will do in the sense that if $K = \{K_1, K_2, \cdots, K_j\}$ then a chain of subsets, $\{\Psi_i\}_{i=1}^{j}$, can be constructed such that $\Psi_i \subset \Psi_{i+1}$ for $i = 1, \cdots, j-1$ and $\Psi_j = X_K$, namely, $\Psi_i \equiv \Psi(\{K_1, \cdots, K_i\})$. Note also that by letting $A_1 = \Psi_1$ and $A_{i+1} = \Psi_{i+1} - \Psi_i$ for $i = 1, \cdots, j-1$, we obtain a partition $(A_1, A_2, \cdots, A_j)$ of X_K. So, if we can characterize the dependencies between A_{i+1} and A_i we will have (perhaps) some information about the cardinality of A_i, $|A_i|$, which will give us at least bounds for $|X_K| = \sum_{i=1}^{j} |A_i|$. Therefore the study of the class of *closed* sets in an *independence system* coming from a closure operator may be reduced to the study of their corresponding kernels. Unfortunately determination of even a single kernel K, for a *closed* set X_K seems to be a hard computational problem because if K and K′ are kernels for X_K and $x \in K$ it is not true, in general, that there exists $y \in K'$ such that $K - \{x\} \cup \{y\}$ is a kernel, so there is not a suitable interchange property based on Ψ (see Williamson [24] for related topics). However, by relaxing the minimality assumption of a kernel and by imposing a mild restriction on each A_i we are able to characterize the elements of A_{i+1}. This is our intention in what follows.

DEFINITION 2.2.

i. A set of triples $O \subseteq T(S_\Sigma)$ is called *realizable* if there exists $A \subseteq S_\Sigma$ such that $T(A) = O$. In this case we will denote $M_A = \Psi(A)$ by M_O.

ii. A set $M = \Psi(A)$ is called *extensible* if there is a transposition $t = (x, y)$ and an element $p \in M$ such that $t \in \tau(p)$, (x and y are adjacent in p), and for all $w \in M$, $w^{-1}(x) < w^{-1}(y)$. In this case we will say that M is *extensible* by the pair (t, p). Note that a set may be *extensible* by many different pairs (t, p).

THEOREM 2:1. *Let* $M \subset S_\Sigma$ *be* extensible *by the pair* (t, p) *where* $t = (x, y)$, $p = uxyv$ *and let* $O = T(M)$. *If* $w \in M_{O \cup T(t(p))}/M_O$ *then* $w = u'yxv'$ *where* $u' \in S_u$, $v' \in S_v$, (S_u *and* S_v *denote the symmetric groups on the synmbols of u and v, respectively*).

Proof. i. First note that because $O \cup T(t(p))$ is a realizable set of triples the notation $M_{O\cup T(t(p))}$ makes sense. $w \in M_{O\cup T(t(p))}/M_O \rightarrow T(w) \cap [T(t(p))/O] \neq \varnothing$ by the definition of Ψ and because $O = T(M)$.

ii. $\varnothing \neq T(w) \cap [T(t(p))/O] \subset T(t(p))/O = \{(-, y, x), (y, x, -)\} \rightarrow$ w cannot be of the form $w = u'xyv'$.

iii. So, w *is of the form* $w = u'yAxv'$ for some $A \subset \Sigma$. The triples in w of the form (y, A, x) (if any) must be in $T(t(p))/O$ because x precedes y in every permutation in M_O by hypothesis. On the other hand $T(t(p))$ does not contain triples of the form (y, A, x) because $t \in \tau(p)$; therefore, $A = \varnothing$ and $w = u'yxv'$.

iv. Suppose now that $u' \notin S_u$ where $p = uxyv$. This means that there exists a symbol $c \in$ symbols of u'/symbols of u and $w = \cdots c \cdots yxv'$, $p = uxy \cdots c \cdots$, $(p) = uyx \cdots c \cdots$.

v. The triple $(c, y, x) \notin O$ because x precedes y in every permutation in M_O, also $(c, y, x) \notin T(t(p))$ by (iv), so $(c, y, x) \notin O \cup T(t(p))$ which means that $w \notin M_{O\cup T(t(p))}$, (contradiction); therefore, symbols of $u' \subset$ symbols of u.

vi. Finally, assume that there exists a symbol c which appears in u but not in u'. We can assume that $w = u'yxv'$ and $t(p) = u' \cdots c \cdots yxv''$ (by v). In this case we have that c appears in v' but not in v'', then $w = u'yx \cdots c \cdots$ and again the triple $(y, x, c) \notin O \cup T(t(p))$, which means that $w \notin M_{O\cup T(t(p))}$, (contradiction); therefore, symbols of $u \subset$ symbols of u'.

(v) and (vi) together give us that if $p = uxyv$ then $w = u'yxv'$ where $u' \in S_u$ and $v' \in S_v$. □

The preceding theorem allows us to express in a very explicit way the relation: between $M_{O \cup T(p)}$ and M_O as stated in the following corollary.

COROLLARY 2.1. *Let* $M \subset S_\Sigma$ *be extensible by* (t, p) *where* t = (x, y), p = *u* *and let* $O = T(M)$. *If* $w \in M_{O \cup T(t(p))}/M_O$ *then* $t^{-1}(w) \in M_O$.

Proof. Let $p = uxyv$ and $t = (x, y)$. $w \in M_{O \cup T(t(p))}/M_O \rightarrow w = u'yxv'$, $u' \in$ $v' \in S_v$ by the preceding theorem. This in turn implies that $T(w)/O = T(t(p))$ and $T(t^{-1}(w))/T(w) \subset T(p)$ because $t^{-1}(w) = u'xyv'$, $u' \in S_u$, $v' \in S_v$; theref $T(t^{-1}(w)) \subset T(p) \cup O = O$, which means that $t^{-1}(w) \in M_O$. □

Corollary 2.1 tells us that the "extension" of a set M by a pair (t, p) is comple determined by a subset of it, namely, $\{q \in M \mid q = u'xyv'$ with $u' \in S_u$, $v' \in S_v$, $p = u$ and $t = (x, y)\}$. Note that the reciprocal of Corollary 2.1 is not true in the sense th: can happen that $t^{-1}(w) \in M_O$ and however $w \notin M_{O \cup T(t(p))}$. This motivates the follov definition.

DEFINITION 2.3. If $M \subset S_\Sigma$ is *extensible* by a pair (t, p), then *the projection s* M with respect to (t, p) will be denoted by $\prod^{M}_{t,p}$ and is defined as follows.

$\prod^{M}_{t,p} \equiv \{q \in M \mid q = u'xyv'$ where $u' \in S_u$, $v' \in S_v$, $p = uxyv$, $t = (x, y)\}$. With definition we have the following corollary.

COROLLARY 2.2. *If* M *is extensible by* (t, p) *and* O = T(M) *then* $M_{O \cup T(t(p}$ $M \cup t(\prod^{M}_{t,p})$.

Proof. The proof follows from Theorem 2.1. and the definition of $\prod^{M}_{t,p}$.

We close this section by mentioning that if X_K is a *closed* set under Ψ and if tl exists a sequence of pairs $\{(t_i, P_i)\}^{j}_{i=1}$, such that $T(K) = \cup^{j}_{i=1} T(P_i)$ and each of sets $\Psi_i = \Psi(\{P_1, \cdots, P_i\})$ is *extensible* by (t_i, P_i) for $i = 1, \cdots, j-1$, then by let $A_1 = \Psi_1$, $A_{i+1} = \Psi_{i+1} - \Psi_i$ for $i = 1, \cdots, j-1$ we obtain a partition $(A_1, \cdots, A_j$ X_K, even though $\{P_i\}^{j}_{i=1}$ is not, in general, a kernel for X_K. All of this is true indepen of the consistency of X_K. In the case that X_K is *consistent* then we can characte algorithmically $\prod^{\Psi_i}_{t_i,p_i}$ for $i = 1, \cdots, j-1$ by looking at the *weak Bruhat* order of This is the purpose of the next section.

3. The weak Bruhat order of S_Σ versus consistent sets.

DEFINITION 3.1.

i. For $u = u_1 \cdots u_n$, let $E(u) = \{(u_i, u_j) \mid i < j, u_i < u_j\}$. E(u) is commonly kn as the set of noninversions of u.

ii. For $\{u, v\} \subset S_\Sigma$ we write,

a) $u \rightarrow v$ if there exists $t \in E(u) \cap \tau(u)$ such that $t(u) = v$.

We say in this case that u *weakly covers* v;

b) $u \dot{\rightarrow} v$ if there exists $t \in E(u)$ such that $t(u) = v$. In this case we say th *strongly covers* v.

iii. The *weak Bruhat* order of S_Σ, β, is defined as follows.

$u \beta v$ if there exists a sequence $(P_0, \cdots, P_m)$, $P_i \in S_\Sigma$ such that $u = P_0$, $P_m = v$ $P_{i-1} \rightarrow P_i$ for $i = 1, \cdots, m$ (Lehmann [19], Savage [21]).

iv. The *strong Bruhat* order of S_Σ, $\dot{\beta}$, is given by $u \dot{\beta} v$ if $u = P_0$, $P_m = v$ $P_{i-1} \dot{\rightarrow} P_i$ for $i = 1, \cdots, m$ (Savage [21], [22]. Clearly $u \beta v \rightarrow u \dot{\beta} v$.

FACT 3.1 (see Fig. 3.1).

i. $u \beta v$ *if and only if* $E(u) \supseteq E(v)$.

ii. *The maps* $f(u) = u \cdot I^R$ *and* $f'(u) = I^R \cdot u$ *are* order reversing involution $\langle S_\Sigma, \beta \rangle$, *i.e.*, $f^2(u) = u$ *and* $u \beta v \rightarrow f(v) \beta f(u)$; *similarly for* $f'(u)$, (I *is the ide* *in* S_Σ, I^R *is its reverse and* $\cdot$ *denotes the usual permutation multiplication*).

iii. $\langle S_\Sigma, \beta \rangle$ *and* $\langle S_\Sigma, \dot{\beta} \rangle$ *are posets with maximum element* I *and minimum eler* I^R. *Moreover* $\langle S_\Sigma, \beta \rangle$ *is a* lattice *by defining the join* $u \vee v$ *of two elements* u *and* *the* minimum element p (*in the weak Bruhat order* β) *such that* $p \beta u$ *and* $p \beta v$ *w*

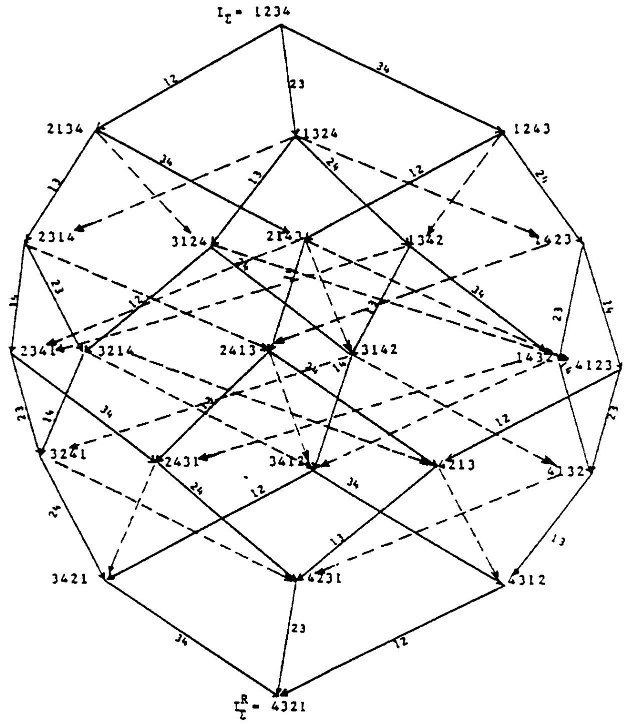

FIG. 3.1. *Bruhat orders on* S_Σ *for* $\Sigma = \{1, 2, 3, 4\}$. *Solid lines denote the covering relations in the weak order and dotted lines correspond to the additional covering relations in the strong ordering. The relevant transpositions are indicated on each edge.*

defining the meet $u \wedge v$ *dually, namely, as the* maximum element p' *such that* $u \beta p'$, $v \beta p'$. *In other words* $u \vee v$ = *least upper bound of* u *and* v *in* β *and* $u \wedge v$ = *greatest lower bound of* u *and* v *in* β.

Proof of i. That $u \beta v$ implies $E(u) \supseteq E(v)$ follows from the definition of β. In the other direction, let j be the minimum i such that $u_i \neq v_i$, (if such j does not exist then $u = v$ and we are done). For this choice of j we have that $u_j < v_j$ ($<$ is the order of Σ) and if $v_j = u_k$ then $u_{k-1} < u_k$ because we are assuming that $E(u) \supseteq E(v)$; therefore, $E(u) \supset E(t(u)) \supseteq E(v)$ where $t = (u_{k-1}, u_k)$. By repeating the argument we construct a chain $u = P_0 \rightarrow \cdots \rightarrow P_m$ with $E(P_m) = E(v)$, so $P_m = v$, which completes the proof. □

Proof of ii. Without loss of generality, take $\Sigma = \{1, 2, \cdots, n\}$. Then we have $f(u) = u \cdot I^R = u^R$, $f'(u) = I^R \cdot u = u'$ with $u'_j = (n + 1) - u_j$ and the result immediately follows. □

Proof of iii. For the proof see Yanagimoto and Okamoto [26].

The following two lemmas give the first relation between the poset $\langle S_\Sigma, \beta \rangle$ and the class of consistent subsets of S_Σ. These results appear in Abello and Johnson [3] and Abello [1], [4] but we reproduce their proofs here for completeness.

LEMMA 3.1. *If* L *is a chain in* $\langle S_\Sigma, \beta \rangle$ *then* L *is a consistent subset of* S_Σ.

Proof (by contradiction). Assume that L is cyclic. Then there are three permutations u, v, w in L and three symbols x, y, z in Σ such that

$$u = \cdots x \cdots y \cdots z \cdots ,$$

$$v = \cdots y \cdots z \cdots x \cdots ,$$

$$w = \cdots z \cdots x \cdots y \cdots .$$

We can assume without loss of generality that $x < y < z$ (the only other essentially different case is $x > y > z$, which can be treated similarly).

i. $E(u)$ contains the ordered pairs (x, y), (x, z), (y, z) and at least two of these pairs do not belong to $E(v)$; thus $E(v) \not\supseteq E(u)$ which means that $v \not\beta u$. Similarly $E(w) \not\supseteq E(u)$ and then $w \not\beta u$. On the other hand $E(v)$ contains (y, z), which does not belong to $E(w)$, then $E(w) \not\supseteq E(v)$, which means $w \not\beta v$.

ii. $E(w)$ contains (x, y), which does not belong to $E(v)$, then $E(v) \not\supseteq E(w)$ and $v \not\beta w$.

(i) and (ii) together give us that v and w are not comparable and therefore u, v, and w cannot be in the same chain (a contradiction). □

Example 3.1. The set $\{1234, 1243, 1423, 4123, 4132, 4312, 4321\}$, which is a subset of $S_{\{1,2,3,4\}}$, is consistent because it is a chain in $\langle S_{\{1,2,3,4\}}, \beta\rangle$ (see Fig. 3.1).

It is interesting to notice that Lemma 3.1 is not true for the *strong Bruhat ordering* $\dot\beta$. For example, $\{2143, 3142, 4321\}$ is a chain in $\langle S_\Sigma, \dot\beta\rangle$; however, it is not *consistent*. This is due to the fact that $\dot\beta$ allows the interchange of nonadjacent elements.

The following is a simple but important property of *maximal* chains in β.

LEMMA 3.2. *If* L *is a maximal chain in* $\langle S_\Sigma, \beta\rangle$ *then* L *is a consistent subset of* S_Σ *such that* $|T(L)| = 4\binom{n}{3}$ and $|L| = \binom{n}{2} + 1$.

Proof. That L is consistent follows from the preceding lemma. Now, $|T(L)| = \binom{n}{3} + \binom{n}{2}(n-2) = 4\binom{n}{3}$ because maximal chains in β have *length* equal to $\binom{n}{2}$. □

The interest of the preceding lemmas is that *for any consistent* set C it must be true that $|T(C)| \leqq 4\binom{n}{3}$ (see Fact 1.1 (iv)) so a maximal chain has the *maximum* number possible of consistent triples; therefore, any *maximal* (with respect to the noncyclicity property) consistent set M which contains a *maximal chain* L must satisfy that $T(M) = T(L)$. Now, if $L = (I = P_0, P_1, \cdots, P_{\binom{n}{2}} = I^R)$ with $t_{i+1}(P_i) = P_{i+1}$ for $i = 0, \cdots, \binom{n}{2} - 1$ and if L_i denotes the unrefinable subchain of L running from I to P_i, i.e., $L_i = \{q \in L, I\, \beta\, q\, \beta\, P_i\}$, then we have that for each i (as above) $\Psi(L_i)$ is a *consistent* set which is *extensible* by the pair (P_i, t_{i+1}) in the sense of § 2; therefore, Theorem 3.2.1 gives important information about the class of *maximal consistent sets* which contain a *maximal* chain in the weak Bruhat order. In fact it provides the basis of an algorithm to construct these sets (Abello [1], [2]).

The preceding ideas carry over to a more general class of consistent sets which contain subsets that are structurally equivalent to chains in the weak Bruhat order. To this end the following definitions are in order.

DEFINITION 3.2.

i. $L \subset S_\Sigma$ is called a *pseudochain* under β if there exists $p \in L$ and a map $m: u \to p^{-1} \cdot u$ such that $m(L)$ is a chain under β. If we want to indicate the dependency between L and p we write $L(p)$ for L. For our purposes any adjectives that apply to chains can be used with pseudochains. Stanley [23] has counted the number of maximal chains, $|C|$, in β; then it follows that the number of maximal pseudochains is $(n!/2)|C|$.

ii. If $L(p)$ is a maximal pseudochain and $m(L) = (I = P_0, \cdots, P_{\binom{n}{2}} = I^R)$ we write $L_i \equiv \{q \in L, I\, \beta\, m(q)\, \beta\, P_i\}$.

iii. For $A \subseteq S_\Sigma$, let $\mathrm{Cov}(A) \equiv \{(p, q) \in A \times A$, p covers q under $\beta\}$ and let λ: $\mathrm{Cov}(S_\Sigma) \to \{(x, y) \in \Sigma \times \Sigma, x < y\}$ be given by $\lambda(p, q) = (x, y)$ if $t(p) = q$ and $t = (x, y)$. λ is called a labelling of the edges in the Hasse diagram of $\langle S_\Sigma, \beta\rangle$. With these conventions let $\mathrm{TRAN}(A) \equiv \lambda(\mathrm{Cov}(A))$.

iv. G_n will denote the undirected (edge labelled) version of the Hasse diagram of $\langle S_\Sigma, \beta\rangle$, namely $G_n = (V, E) = (S_\Sigma, \mathrm{Cov}(S_\Sigma))$ where the edge $(p, t(p))$ is labelled by the two subset $\{x, y\}$ if $t = (x, y)$.

The following lemma states the equivalence between chains and pseudochains from the consistency point of view and it identifies pseudochains in $\langle S_\Sigma, \beta\rangle$ with shortest paths in G_n.

LEMMA 3.3. *Let* $L(p)$ *be a pseudochain in* $\langle S_\Sigma, \beta\rangle$.

i. $\Psi(L(p))$ *is a consistent subset of* S_Σ (*see definition of* Ψ *in* § 2).

ii. *If* $t, l \in \mathrm{TRAN}(L(p))$ *then* $t \neq l$ *and* $t^{-1} \neq l$ (*if* $t = (x, y)$, $t^{-1} = (y, x)$).

iii. $L(p)$ *is a* saturated (*unrefinable*) *pseudochain from* p *to* q *if and only if* $L(p)$ *is a* shortest path *from* p *to* q *in* G_n.

iv. *If* SPATH (p, q) *denotes a shortest path from* p *to* q *in* G_n *then* SPATH (p, q) *is consistent*.

Proof. For (i) note that $L(p)$ is consistent because it is the image of a chain in β under a uniform relabelling, m, of the symbols of Σ, and chains in β are consistent by Lemma 3.1; therefore, $\Psi(L(p))$ is consistent.

For (ii) and (iii) note that if $p = p_1 p_2 \cdots p_n \in S_\Sigma$ and $t = (p_i, p_{i+1})$ then $t(p) = p \cdot l(I)$ where $l = (i, i+1)$. Now, left multiplication by a fixed permutation is an automorphism of S_Σ that preserves adjacency in the weak Bruhat order (for example, $p \rightarrow p^{-1} \cdot p = I$ and $t(p) \rightarrow p^{-1} \cdot t(p) = l(I)$); therefore, it does preserve distances. In particular a shortest path SPATH (p, q) is mapped by left multiplication to SPATH $(I, p^{-1} \cdot q)$. But shortest paths, in G_n, from the identity I to any permutation w are saturated chains in β. This can be seen by induction on the path length which is nothing else than the number of inversions of w.

(iv) is just the result of putting (i) and (iii) together. □

The preceding lemma will allow us to state consistency results in terms of shortest paths in G_n even if we give proofs of them only in terms of chains in $\langle S_\Sigma, \beta\rangle$.

The following result gives information about certain subconfigurations of any consistent subset M of S_Σ. Note that no assumptions are made about the *connectivity* (in the graph sense) or *maximality* of M.

LEMMA 3.4. *Let* M *be a* consistent *subset of* S_Σ, $q \in M$, $p \in S_\Sigma$ *and let* SPATH (p, q) *and* SPATH′ (p, q) *be two different shortest paths from* p *to* q *such that* $t(p) \in$ SPATH (p, q), $t'(p) \in$ SPATH′ (p, q) *where* t *and* t′ *are two different adjacent transpositions* (*see* Fig. 3.2 *below*). *Under these conditions*, $\{t(p), t'(p)\} \subset M \rightarrow t \cap t' = \emptyset$.

Proof (by contradiction). (i) Assume that $t \cap t' \neq \emptyset$ and without loss of generality let $t = (x, y)$, $t' = (y, z)$ and suppose that SPATH (p, q) and SPATH′ (p, q) are chains in $\langle S_\Sigma, \beta\rangle$. With these assumptions q becomes a *lower bound* for $t(p)$ and $t'(p)$ which means that the set of inversions of q, INV (q), contains INV (t(p)) ∪ INV (t′(p)); therefore, INV (q) ⊃ {(y, x), (z, y)}, which implies that $(z, y, x) \in T(q)$ because SPATH and SPATH′ are shortest paths.

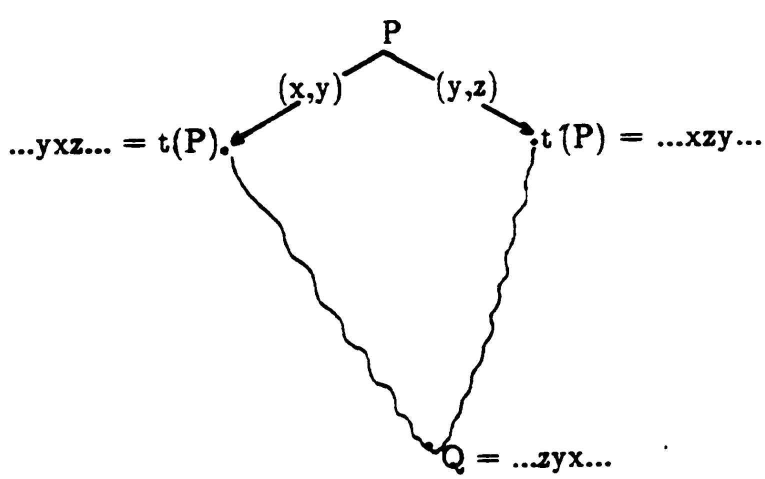

FIG. 3.2. *Illustration of Lemma* 3.4. *Note that* P *is not required to be in* M.

(ii) On the other hand, the fact that $t \cap t' \neq \emptyset$ forces $(y, x, z) \in T(t(p))$ and $(x, z, y) \in T(t'(p))$. (i) and (ii) together contradict the *consistency* of M. □

The fact that $\langle S_\Sigma, \beta \rangle$ is a lattice (Fact 1.iii) gives us the following corollary as a special case.

COROLLARY 3.1. *Let* $\{q, w, v\} \subset M \subset S_\Sigma$ *and let* t, t' *be two different adjacent transpositions.*

i. *If* $t(w \vee v) = w$, $t'(w \vee v) = v$, $w \beta q$, $v \beta q$ *and if* M *is consistent then* $t \cap t' = \emptyset$.

Dually we have,

ii. *If* $t(w) = w \wedge v$, $t'(v) = w \wedge v$, $q \beta w$, $q \beta v$ *and if* M *is consistent then* $t \cap t' = \emptyset$.

Proof. (i) and (ii) follow from the preceding lemma by taking $p = w \vee v$ and $p = w \wedge v$, respectively. □

Maximal consistent subsets in the *weak Bruhat order* exhibit a "local semimodularity" property which does not hold for the *strong Bruhat order*. This is stated precisely in the following corollary whose content will be referred to as *the Quadrilateral rule* or *the* Q *rule*.

COROLLARY 3.2 (the Quadrilateral rule). *Let* M *be a consistent subset of* S_Σ *and* $\{w, v\} \subset M$. *If there exist* $\{p, q\} \subset S_\Sigma$ *and two different adjacent transpositions* t *and* l *such that* $l(w) = q = t(v)$ *and* $t^{-1}(w) = p = l^{-1}(v)$ *then* $\{w, v, p, q\} \subset \Psi(M)$ *(see* Fig. 3.3).

Proof. The conditions imposed to l and t in the hypothesis hold if and only if $l \cap t = \emptyset$ and this in turn implies that $T(\{p, q\}) = T(\{w, v\}) \subset T(M)$; therefore, $\{p, q, w, v\} \subset \Psi(M)$ (this is not true if t and l are not adjacent transpositions and then it is not true in the *strong Bruhat* order). □

In terms of the weak Bruhat order, the Q rule says that for any two elements w, v of a maximal consistent set $\Psi(M)$, if their join, $w \vee v$, covers both w and v *and* if their meet, $w \wedge v$, is covered also by *both* w and v then $\{w, v, w \vee v, w \wedge v\} \subset \Psi(M)$. This resembles the definition of an Upper Semimodular lattice (Birkhoff [8]). However, the problem here is that *both* conditions $w \vee v \rightarrow \{w, v\}$ and $\{w, v\} \rightarrow w \wedge v$ are necessary, neither one implies the other, and moreover it is not true in general that $\Psi(M)$ is even a sublattice of $\langle S_\Sigma, \beta \rangle$. On the other hand, if M is a chain in β then $\Psi(M)$ is not only a sublattice but an upper semimodular one as will be established in Theorem 3.3.

The following result is basically an iterated application of the Quadrilateral rule.

THEOREM 3.1. *Let* M *be a* consistent *subset of* S_Σ *and let* $p, q \in \Psi(M)$ *such that* $p = uxyv$, $q = u'xyv'$ *where* $u' \in S_u$, $v' \in S_v$. *If there exists a shortest path* SPATH (q, p) $\subset \Psi(M)$ *such that for all* $w \in$ SPATH (q, p), $w^{-1}(x) < w^{-1}(y)$ *then for all* $w \in$ SPATH (q, p), $w = u''xyv''$ *where* $u'' \in S_u$, $v'' \in S_v$.

Proof (by induction on |SPATH (q, p)|).

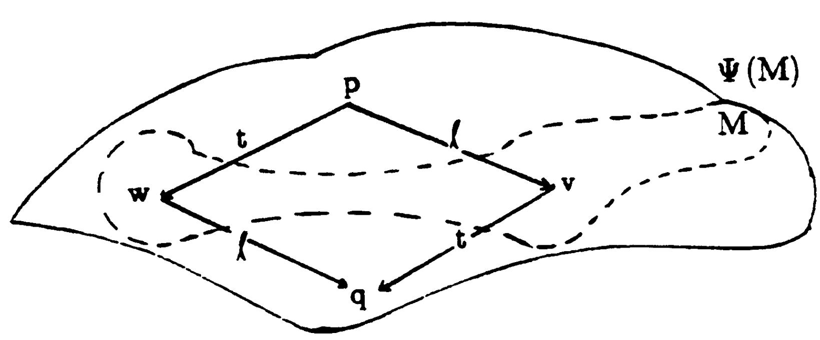

FIG. 3.3. *The Quadrilateral rule.*

Notation. If $p \in S_\Sigma$ and $a \in \Sigma$, denote by $p/_a$ the permutation in $S_{\Sigma-\{a\}}$ obtained by erasing a from p.

Basis. If $|SPATH(q, p)| = 1$ then there is nothing to prove.

(i) *Induction Hypothesis*. Assume it is true for $|SPATH(q, p)| = j \leqq k < \binom{|\Sigma|}{2}$ and let $|SPATH(q, p)| = k + 1$. Let $w' \in SPATH(q, p)$ and $l'(q) = w'$ where $l' \in TRAN(SPATH(q, p))$ and assume that $l' \cap \{x, y\} \neq \varnothing$. Without loss of generality let $l' = (a, x)$. By assumption $u' \in S_u$ and therefore a must precede x in p; therefore, there exists $l \in TRAN([w', p])$ such that $l = (x, a)$, ([w, p] denotes the subpath of SPATH (q, p) running from w down to p). Take the first such l in TRAN ([w', p]) and let w be the permutation in SPATH (q, p) to which l is applied, so $w = u''xav''$ and $w' = (u'/_a)xa(yv')$. Assume now that there exists $c \in u''$ such that $c \notin u'/_a$, so $c \neq a$ because $a \notin u''$ and $c \neq y$ because $w^{-1}(x) < w^{-1}(y)$ by hypothesis; therefore, $(c, x, a) \in T(w)$, $(x, a, c) \in T(w')$, which imply that $(c, a, x) \in T(l(w))$ and $(a, x, c) \in T(p) \cap T(q)$. This forces [w', w] to contain a permutation w″ which contains the triple (x, c, a) because to go from w′ to w, a and c must be interchanged without interchanging (x, a) by the choice of l, and for c to precede x in w, at some point in [w′, w], c must be between x and a (the preceding argument depends exclusively on the connectivity of SPATH (q, p) and on the choice of $l = (x, a)$). Therefore, $\{w'', l(w), p\}$ contains a cyclic triple, namely, $\{(x, c, a), (c, a, x), (a, x, c)\}$ *contradicting* the consistency of M. Up to this point we have proved that symbols of $u'' \subset$ symbols of $u'/_a$ and by a symmetric argument we obtain that symbols of $u'/_a \subset$ symbols of u'', which means that $u'' \in S_{u'/_a}$, $w = u''xav''$, $w' = (u'/_a)xa(yv')$; therefore, the subpath [w′, w] has length $|[w', w]| \leqq k$ and satisfies the hypothesis of the theorem, so by Induction Hypothesis every permutation on it is of the form $u'''xav'''$ with $u''' \in S_{u'/_a}$, $v''' \in S_{yv'}$, and if $t \in TRAN([w', w])$ then $t \cap l = \varnothing$.

(ii) Now, the maximality of $\Psi(M)$, the fact that $[w', w] \subset \Psi(M)$, and (i) allow us to apply iteratively *the Quadrilateral rule* to get that $l([w', w]) \subset \Psi(M)$, giving us that the path $(q, l(w'), l([w', w]), [l(w), p])$ is a path from q to p that is shorter than SPATH (q, p), which is a contradiction; therefore, the original assumption that $l \cap \{x, y\} \neq \varnothing$ was false.

By (ii), $l \cap \{x, y\} = \varnothing$ and then $l(q)$ and p satisfy the hypothesis of the theorem, and by induction we will be done. □

Theorem 3.1, coupled with the results of § 2, gives the following characterization of *extensible consistent* subsets of S_Σ.

THEOREM 3.2 (see § 2 for related definitions). *Let* M *be a* consistent *subset of* S_Σ *which is also* extensible *by a pair* (t, p) *and let* $w \in t(\prod^M_{t,p})$. *If there exists a shortest path* $SPATH(t^{-1}(w), p) \subset \Psi(M)$ *and if* $l \in TRAN(SPATH(t^{-1}(w), p))$ *then* $l \cap t = \varnothing$. (*We will refer to this theorem as* the projection theorem).

(i) *Proof.* If $w \in t(\prod^M_{t,p})$ then $t^{-1}(w) \in \Psi(M)$ by Corollary 2.1 and by the definition of $\prod^M_{t,p}$.

Now, $p \in \prod^M_{t,p}$ and $SPATH(t^{-1}(w), p) \subset \Psi(M)$ satisfy the hypothesis of Theorem 3.1 because M is an *extensible consistent* subset of S_Σ; therefore, $SPATH(t^{-1}(w), p) \subset \prod^M_{t,p}$ which means that $l \cap t = \varnothing$ for every $l \in TRAN(SPATH(t^{-1}(w), p))$. □

The preceding theorem tells us that within each connected component of an *extensible* set, which is also *consistent*, the elements of $\prod^M_{t,p}$ *are precisely* those that are connected by paths all of whose transpositions are disjoint from t.

A lattice semimodular property of consistent sets. Recall that a lattice **L** is upper semimodular if it satisfies the following condition:

The U.S. Condition: For all elements w and v of **L** if w covers $w \wedge v$ then $w \vee v$ covers v. The following seemingly weaker condition is sufficient to prove upper semimodularity (Birkhoff [8]):

The **W.U.S.** Condition: For all elements w and v of **L**, if $w \wedge v$ is covered by both w and v then $w \vee v$ must cover both w and v.

As another application of the Q-rule we have the following result.

LEMMA 3.5. *Let* M *be a consistent subset of* $\langle S_\Sigma, \beta\rangle$. *If* $\Psi(M)$ *is a meet subsemilattice (join subsemilattice) of* $\langle S_\Sigma, \beta\rangle$ *with a maximum element (minimum element) then* $\Psi(M)$ *is an* upper semimodular sublattice *of* $\langle S_\Sigma, \beta\rangle$.

Proof. That $\Psi(M)$ is a meet sublattice with a maximum element automatically implies that $\Psi(M)$ is a lattice.

To prove that $\Psi(M)$ is upper semimodular is enough to prove that $\Psi(M)$ satisfies the **W.U.S.** condition. To this end let w and $v \in \Psi(M)$, $w \wedge v \in \Psi(M)$. Now let q be some upper bound for both v and w and assume that there are adjacent transpositions t and t′ such that $t(w) = w \wedge v$, $t'(v) = w \wedge v$ (i.e., $w \wedge v$ is covered by both v and w). The consistency of $\Psi(M)$ allows us then to apply Corollary 3.1 (ii) to conclude that $t \cap t' = \emptyset$, which in turn implies by the quadrilateral rule that the element $w \vee v = t^{-1}(v) \in \Psi(M)$ satisfies that $t'(w \vee v) = w$. This proves that $w \vee v$ covers both w and v which is the conclusion of the **W.U.S.** condition. □

Notation. For the remainder of this section we will follow the following notational conventions.

i. *Ch* will always denote a saturated chain (or pseudochain) $Ch = (P_0, P_1, \cdots, P_k)$ where $t_{i+1}(P_i) = P_{i+1}$ for $i = 0, \cdots, k-1$.

ii. $[P_0, P_i] \equiv \{p \in Ch \mid P_0 \,\beta\, p \,\beta\, P_i\}$; $Ch_i \equiv \Psi([P_0, P_i])$.

The following basic properties of the weak Bruhat order will be instrumental in the proof of the main result of this section.

LEMMA 3.6. *For* $p \in S_\Sigma$ *consider the set* $E(p)$ *of noninversions of* p *as a binary relation on* Σ *and denote by* $(E(p))^*$ *its transitive closure. With these conventions, we have*:

i. $p \vee q$ *is the unique permutation satisfying that* $E(p \vee q) = (E(p) \cup E(q))^*$;

ii. *If* $p = uxyv$ *and* $q = u'xyv'$ *where* $x < y$, u *and* u′ *in* S_{Σ_1}, v *and* v′ *in* S_{Σ_2}, *then* $p \vee q = (u \vee u')xy(v \vee v')$;

iii. *If* $t = (x, y) \in E(p) \cap E(q)$ *and if* t *is an admissible transposition of* p *then* $p \vee q = t(p) \vee q$.

Proof.

i. For the proof, see Berge [7].

ii. Note that $E(p)$ and $E(q)$ differ only in $E(u)$, $E(u')$, $E(v)$, and $E(v')$, respectively. This forces $(E(p) \cup E(q))^*$ to be equal to $E((u \vee u')xy(v \vee v'))$, which together with (i) implies that $p \vee q = ((u \vee u')xy(v \vee v'))$.

iii. The fact that $(x, y) \in E(p) - E(t(p))$, $E(t(p)) \subset E(p)$, and $(x, y) \in E(q)$ implies that $E(t(p)) \cup E(q) = E(p) \cup E(q)$ and again by (i), $t(p) \vee q = p \vee q$. □

Theorem 3.2 (the projection theorem) and the Q-rule, together with the fact that $[P_0, P_i]$ is a saturated chain (or pseudochain) imply that $Ch_i = \Psi([P_0, P_i])$ is a connected subset of S_Σ.

Now, if $i = 1$, $\Psi([P_0, P_i]) = (P_0, P_1)$, which is clearly a join sublattice with top element P_0. For the general case note that $Ch_{k+1} - Ch_k = t_{k+1}(\prod^{Ch_k}_{t_{k+1}, P_k})$ by Corollary 2.2. But this is saying that $Ch_{k+1} - Ch_k$ is obtained from $\prod^{Ch_k}_{t_{k+1}, P_k}$ by right multiplication by a fixed permutation, namely the one corresponding to the transposition t_{k+1}. Moreover, if two elements are adjacent in $\prod^{Ch_k}_{t_{k+1}, P_k}$, their images under t_{k+1} must be adjacent. So we have here a one-to-one mapping that preserves adjacencies and therefore distances under β. Therefore, if $v, w \in Ch_{k+1} - Ch_k$ then $t^{-1}_{k+1}(w)$ and $t^{-1}_{k+1}(v) \in \prod^{Ch_k}_{t_{k+1}, P_k}$, and by Lemma 3.6 (ii) we can assume that $z = t^{-1}_{k+1}(w) \vee t^{-1}_{k+1}(v) \in \prod^{Ch_k}_{t_{k+1}, P_k}$, which allows us to conclude that $t_{k+1}(z) = w \vee v \in Ch_{k+1} - Ch_k$. If $v \in Ch_k$ and $w \in Ch_{k+1} - Ch_k$, then the fact that Ch_k is extensible by (t_{k+1}, P_k) allows us to apply Lemma 3.6 (iii) by letting $q = v$ and $p = t_{k+1}^{-1}(w)$ to obtain that $v \vee w \in Ch_{k+1}$.

The preceding arguments show that Ch_i is a sublattice of $\langle S_\Sigma, \beta \rangle$ with top element, and therefore by Lemma 3.5 we have the following promised result.

THEOREM 3.3. *If* M *is a saturated chain in the weak Bruhat order then* $\Psi(M)$ *is an upper semimodular sublattice of* $\langle S_\Sigma, \beta \rangle$.

Remarks. The preceding results play a central role in the algorithmic construction of maximal consistent sets which contain a saturated chain (or pseudochain) Ch in $\langle S_\Sigma, \beta \rangle$. It says that if $Ch_i \equiv \Psi([P_0, P_i])$ has been constructed then to find $\prod_{t_{i+1},P_i}^{Ch_i}$ one backtracks (in Ch_i) from P_i by following any path whose transpositions are disjoint from t_{i+1}. At every step all that is required is to find one incoming transposition l disjoint from t_{i+1}. Theorem 3.3 guarantees that the process will stop if and only if at some point we reach one permutation all of whose incoming transpositions intercept t_{i+1} (the formal algorithm can be found in Abello [1], [4], where it is called the MCCS algorithm).

4. Weak Bruhat order, consistent sets and Catalan numbers. We will prove here that the nth Catalan number is an upper bound for those consistent sets containing a Maximal pseudochain in the weak Bruhat order.

DEFINITION 4.1.

i. If M is a connected subset of S_Σ, its diameter, diam (M), is defined as diam (M) $\equiv \max_{\{P,Q\} \subset M} |\mathrm{SPATH}(P, Q)|$.

ii. For a saturated chain (pseudochain) $Ch = [P, Q]$ in $\langle S_\Sigma, \beta \rangle$ denote by OTRAN (Ch) the *ordered* set of transpositions used in Ch, namely OTRAN $(Ch) \equiv \{t_i\}_{i=|Ch|-1}$ where $t_{i+1}(P_i) = P_{i+1}$, $P_i \in Ch$; and let Ch^x be the *subsequence* of OTRAN (Ch) consisting of transpositions involving $x \in \Sigma$. Elements of Ch^x will be distinguished by having a superscript x, namely, $Ch^x = (t_1^x, t_2^x, \cdots, t_j^x)$.

iii. $[l_j, l_k] \equiv \{l_i \in \mathrm{OTRAN}(Ch)$ such that $j \leqq i \leqq k\}$.

iv. For a *subsequence* $(l_1, l_2, \cdots, l_j)$ of Ch^x and a permutation Q, we will write $(l_j, \cdots, l_1)(Q)$ to denote the *sequence* of permutations $(Q = Q_0, Q_1, \cdots, Q_j)$ where $Q_{i+1} = l_i(Q_i)$ for $i = 1, \cdots, j-1$.

The following is a technical lemma that will allow us to single out a very special canonical subchain in M_{Ch}.

FACT 4.1. *Assume that* $[p, v]$ *is a saturated chain in* $\langle S_\Sigma, \beta \rangle$ *such that* $p_1 = v_i = x \in \Sigma$ *and* $p_n = v_{i+1} = y \in \Sigma$ *and let us recall that if* $p \in S_\Sigma$, $\tau(p)$ *denotes its admissible set of transpositions. If* $t_q, t_r \in \mathrm{OTRAN}([p, v])$ *are such that* $t_q = t_1^x = (x, a)$, $t_r = t_2^x = (x, b)$, $a \neq y$, $b \neq y$ *with* $t_q \in \tau(Q)$, $t_r \in \tau(R)$ *and* $[Q, R] \subset [p, v]$ *then* $M_{[p,R]} = M_{[p,Q] \cup (t_q, t_{r-1}, \cdots, t_{q+1})(Q)}$. *We will say in this case that the sequence* $(t_{q+1}, \cdots, t_{r-1})$ *has been lifted by the transposition* t_q *(see Fig.* 4.1).

Proof. $M_{[p,R]} = M_{[p,Q] \cup (t_{r-1}, \cdots, t_{q+1}, t_q)(Q)}$ by the definition of t_q and t_r.

(i) $t_r = t_2^x \rightarrow$ each transposition in $(t_{q+1}, \cdots, t_{r-1})$ does not involve x.

(ii) $t_q = t_1^x$ and the assumption that $[Q, R]$ is a chain $\rightarrow$ each transposition in $(t_{q+1}, \cdots, t_{r-1})$ does not involve the symbol a.

Therefore, the Quadrilateral rule (Corollary 3.2), can be applied (iteratively) to $(t_{q+1}, \cdots, t_{r-1})$ by (i) and (ii) and the result follows by the maximality of $M_{[p,R]}$. □

Remark. The idea of lifting one sequence, by one transposition (Fact 4.1), can be used iteratively, in certain cases, to lift one sequence by another as follows. Consider two permutations p and q such that $p \,\beta\, q$, $p_j = q_j = x$ and assume that there is a saturated chain Ch from p to q such that if $t = (a, b) \in \mathrm{OTRAN}(Ch)$ then $a \neq x \neq b$. Now, let LEFT $(Ch) = (t_{i_1}, \cdots, t_{i_k})$ denote the subsequence of OTRAN (Ch) obtained by deleting from it those transpositions using symbols in $\{p_1, \cdots, p_{j-1}\}$. Simi-

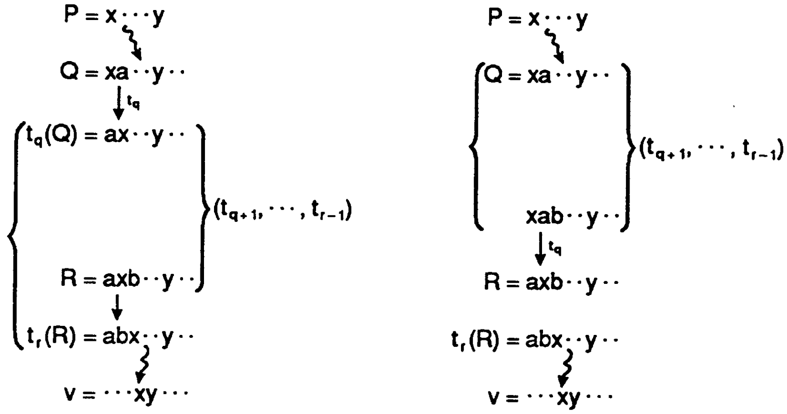

FIG. 4.1. *The lifting of a sequence* $(t_{q+1}, \cdots, t_{r-1})$ *by a transposition* t_q.

larly, let RIGHT $(Ch) = (t_{j_1}, \cdots, t_{j_{k'}})$ denote the subsequence of OTRAN (Ch) obtained by deleting from it those transpositions using symbols in $\{p_{j+1}, \cdots, p_k\}$. (For our purpose assume that both LEFT (Ch) and RIGHT (Ch) are nonempty and that the last transposition of OTRAN (Ch) is an element of RIGHT (Ch)). Note that if $t \in$ LEFT (Ch) and $t' \in$ RIGHT (Ch) then $t \cap t' = \emptyset$. This together with the assumption that Ch is a saturated chain in β all of whose elements have the symbol x exactly in the same position implies that the sets of permutations $(t_{i_k}, \cdots, t_{i_1})(p)$ and $(t_{j_{k'}}, \cdots, t_{j_1})(p)$ are saturated chains in $\langle S_\Sigma, \beta\rangle$. This can be seen by an iterated lifting of certain subsegments of the sequence LEFT (Ch) by each of the elements of RIGHT (Ch) (in reverse order) in an iterated fashion. The figure below illustrates this process for the case where RIGHT (Ch) consists of two transpositions only. Note that because here we use only the Quadrilateral rule, then the set of ordered triples of $(t_{i_k}, \cdots, t_{i_1})(p)$, $T((t_{i_k}, \cdots, t_{i_1})(p))$, together with the set of ordered triples of $(t_{j_{k'}}, \cdots, t_{j_1})(p)$, $T((t_{j_{k'}}, \cdots, t_{j_1})(p))$, is precisely equal to the set of ordered triples of Ch, $T(Ch)$.

Note that because the process depicted in Fig. 4.2 consists of repeated applications of the Quadrilateral rule, we can be sure that all the saturated chains Ch' from p to q that are obtained in this manner satisfy that $T(Ch') = T(Ch)$ which means that $Ch' \subset \Psi(Ch)$. In particular this is true for the chain determined by using first (in order) the transpositions of LEFT (Ch) and then the transpositions of RIGHT (Ch), which in our unwanted (very clumsy) notation is denoted by $((t_{j_{k'}}, \cdots, t_{j_1})(t_{i_k}, \cdots, t_{i_1}))(p)$.

We collect the preceding remarks and the process depicted in Fig. 4.2 in the following result.

FACT 4.2. Let p, q be permutations in S_Σ that satisfy p β q, $p_j = q_j = x$ and let Ch denote a saturated chain from p to q such that if $t = (a, b) \in$ OTRAN (Ch) then $a \neq x \neq b$. Under these conditions it is possible to find a saturated chain Ch' from p to q such that:

i. OTRAN (Ch') consists first of all tranpositions in OTRAN (Ch) which use only symbols in $\{p_{j+1}, \cdots, p_n\}$ (call this set LEFT (Ch)) followed by all transpositions in OTRAN (Ch) using only symbols in $\{p_1, \cdots, p_{j-1}\}$ (call this set RIGHT (Ch)) (or vice versa). In symbols: OTRAN (Ch') = (LEFT (Ch), RIGHT (Ch)) or OTRAN (Ch') = (RIGHT (Ch), LEFT (Ch)).

ii. $T(Ch') = T(Ch)$ or equivalently $Ch' \subset \Psi(Ch)$.

iii. (a) If RIGHT $(Ch) = (t_{j_1}, \cdots, t_{j_{k'}})$ then all the permutations in the set $(t_{j_{k'}}, \cdots, t_{j_1})(p)$ have as a common suffix the subpermutation $p_{j+1} \cdots p_n$. By deleting this common suffix from all of them we obtain a saturated pseudochain in

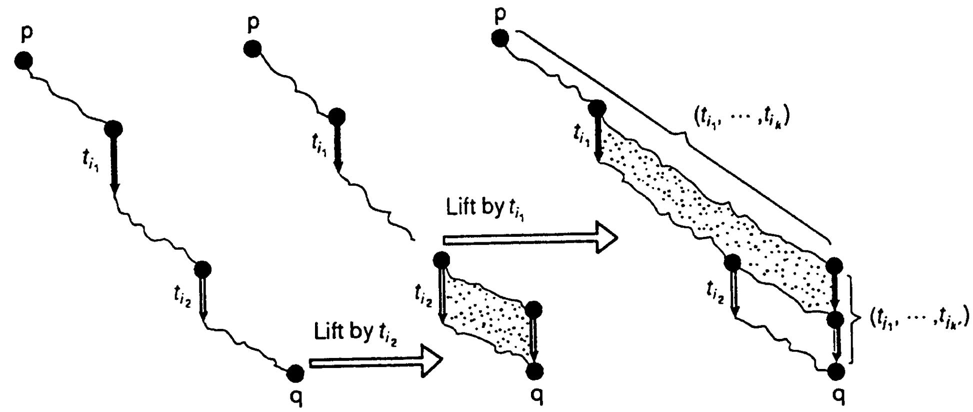

FIG. 4.2. *Lifting of a sequence* LEFT(Ch) *by another sequence* RIGHT$(Ch) = (t_{i_1}, t_{i_2})$. *This assumes that all the elements in the chain Ch from* p *to* q *contain a fixed symbol x in exactly the same position.*

$\langle S_{\{p_1, \cdots, p_{j-1}\}}, \beta\rangle$ from $p_1 \cdots p_{j-1}$ to $q_1 \cdots q_{j-1}$. Call this pseudochain RESTRICTED_RIGHT (Ch) and its closure FIRST_HALF $\Psi(Ch)$.

(b) If LEFT $(Ch) = (t_{i_1}, \cdots, t_{i_k})$ then all the permutations in the set $(t_{i_k}, \cdots, t_{i_1})(p)$ have as a common prefix $p_1 \cdots p_{j-1}$. By deleting this common prefix from all of them we obtain a saturated pseudochain in $\langle S_{\{p_{j+1}, \cdots, p_n\}}, \beta\rangle$ from $p_{j+1} \cdots p_n$ to $q_{j+1} \cdots q_n$. Call this pseudochain RESTRICTED_LEFT (Ch) and its closure SECOND_HALF $\Psi(Ch)$.

As a justification (if any) for the definitions given in (a) and (b) above we have the following:

(c) For a chain Ch satisfying the restrictions given above we have that $\Psi(Ch) =$ FIRST_HALF $(\Psi(Ch)) \times [x] \times$ SECOND_HALF $(\Psi(Ch))$, (here $\times$ denotes cross product).

Note. Everything we have discussed after Fact 4.1 is put very concisely in the following definition and theorem. However, if the reader feels comfortable he/she may jump directly to the remarks preceding Theorem 4.2 without losing continuity.

DEFINITION 4.2.

i. For $(t_{i_1}, t_{i_2}, \cdots, t_{i_j})$ a subsequence of OTRAN $([P, P^R])$ such that $(t_{i_1}, t_{i_2}, \cdots, t_{i_j}) = (t_1^x, t_2^x, \cdots, t_j^x)$ denote by $\{Q_l\}_{l=1}^{j}$ the subchain of $[P, P^R]$ such that $t_l^x \in \tau(Q_l)$.

ii. Let LEFT (t_l^x, t_{l+1}^x) denote the subsequence of $[t_{i_l}, t_{i_{l+1}}]$ obtained by deleting from it those transpositions using symbols that precede x in Q_l. Similarly, let RIGHT (t_l^x, t_{l+1}^x) denote the subsequence of $[t_{i_l}, t_{i_{l+1}}]$ obtained by deleting from it those transpositions using symbols that follow x in Q_l.

iii. Let TRANSFORM $(t_l^x, t_{l+1}^x) \equiv$ (RIGHT (t_l^x, t_{l+1}^x), LEFT (t_l^x, t_{l+1}^x), t_l^x) and TRANSFORM $(t_l^x, t_j^x) \equiv$ (TRANSFORM (t_l^x, t_{l+1}^x), TRANSFORM (t_{l+1}^x, t_{l+2}^x), $\cdots$, TRANSFORM (t_{j-1}^x, t_j^x)).

The following result is just an iterated application of Fact 4.1 in which a sequence was lifted by one transposition. In the following theorem a sequence is lifted by another sequence.

THEOREM 4.1. *If* $(t_{i_1}, t_{i_2}, \cdots, t_{i_j}) = (t_1^x, t_2^x, \cdots, t_j^x)$ *with* $t_1^x \in \tau(Q)$, $t_j^x = t_s \in \tau(S)$ *and* $[Q, S] \subset [p, v]$ *then* $M_{[p,S]} = M_{[p,Q] \cup (t_j^x, \text{TRANSFORM}(t_1^x, t_j^x))(Q)}$.

Proof. (By induction on j).

Basis. If $j = 2$, the result follows from Fact 4.1.

Induction Hypothesis. Assuming the result is true for j, we will prove it for $j + 1$.

Suppose $(t_{i_1}, \cdots, t_{i_j}, t_{i_{j+1}}) = (t_1^x, t_2^x, \cdots, t_j^x, t_{j+1}^x)$ and let $t_{i_j} \in \tau(R)$, $t_{i_{j+1}} \in \tau(u)$ with $[p, R] \cup [R, u] \subset [p, v]$. By Induction Hypothesis $M_{[p,R]} =$

$M_{[p,Q]\cup(t_j^x,\text{TRANSFORM}(t_1^x,t_j^x))(Q)}$. By definition of t_{j+1}^x every transposition in OTRAN $[t_{i_j}(R), u]$ does not use the symbol x. This implies that the quadrilateral rule may be applied to OTRAN $[t_{i_j}(R), u]$ to lift the transpositions in LEFT (t_j^x, t_{j+1}^x) by those transpositions in RIGHT $(t_j^x, t_{j+1}^x)/t_{j+1}^x$. But this means that instead of OTRAN $[t_{i_j}(R), u]$ we may use (RIGHT (t_j^x, t_{j+1}^x), LEFT (t_j^x, t_{j+1}^x)). Therefore, $M_{[p,u]} = M_{[p,Q]\cup(t_{j+1}^x,\text{RIGHT}(t_j^x,t_{j+1}^x),\text{LEFT}(t_j^x,t_{j+1}^x),(t_j^x,\text{TRANSFORM}(t_1^x,t_j^x)))(Q)}$ by Induction Hypothesis and by the maximality of $M_{[p,u]}$. Now, by noticing that the right-hand side of the last equation is equal to $M_{[p,Q]\cup(t_{j+1}^x,\text{TRANSFORM}(t_1^x,t_{j+1}^x))}$ the result follows. □

Remarks. We have seen that a shortest path SPATH (p, q) is mapped bijectively to a saturated chain *Ch* in $\langle S_\Sigma, \beta\rangle$ by left multiplication by p^{-1}. This induces a map from the ordered triples of SPATH (p, q), T(SPATH (p, q)), to the ordered triples of *Ch*, T(*Ch*); namely, if R ∈ SPATH (p, q), (x, y, z) ∈ T(R) if and only if $(p^{-1}(x), p^{-1}(y), p^{-1}(z)) \in T(p^{-1}\cdot R)$. But this means that $w \in \Psi(\text{SPATH}(p, q))$ if and only if $p^{-1}\cdot w \in \Psi(Ch)$; therefore, $|\Psi(\text{SPATH}(p, q))| = |\Psi(Ch)|$. Therefore, for every *maximal connected consistent set* (m.c.c.s.) $M \subset S_\Sigma$ of diameter $\binom{n}{2}$ where $n = |\Sigma|$ there exists a m.c.c.s. $M' \subset S_\Sigma$ that contains a maximal chain such that $|M| = |M'|$. This is not saying that all such sets (with the same diameters) have the same cardinality (in fact their cardinalities are in general quite different as proved in Abello [2–4]). With this in mind we will denote by M_j any maximal connected consistent subset of S_Σ where $|\Sigma| = j$. Now if M_j has diameter $\binom{j}{2}$ we may assume that it contains a maximal chain under β.

Finally, we will prove the next result which relates *Catalan numbers* and *maximal connected consistent sets*.

THEOREM 4.2. *For* $|\Sigma| = n$. *If* M_n *denotes a maximum connected consistent subset of* $\langle S_\Sigma, \beta\rangle$ *of diameter*, diam $(M_n) = \binom{n}{2}$ *then* $\langle M_n, \beta\rangle$ *is an upper semimodular lattice with cardinality* $|M_n| < (1/n+1)\binom{2n}{n}$ = *the* nth *Catalan number* C_n *for* $n > 2$.

Proof. The upper semimodularity of $\langle M_n, \beta\rangle$ was established in the preceding section (Theorem 3.3), so we will prove here that $|M_n| \leqq C_n$.

For simplicity in notation we will write $\prod_t^B$ to denote the projection set $\prod_{t,P}^B$ of *B* with respect to (t, P), if there is no danger of confusion.

(i) By the remarks preceding this theorem we may assume that M_n contains a maximal chain $Ch = [I, I^R]$ in β. Let $I_1 = x \in \Sigma$ and $I_n = y \in \Sigma$. By noting that x never moves to the left in *Ch* we have that OTRAN $(Ch) = (t_1, \cdots, t_{\binom{n}{2}})$ imposes a total order < on $\Sigma - x$ given by $b_i < b_j$ if and only if $t_i = (x, b_i)$, $t_j = (x, b_j)$ and $i < j$.

(ii) Now, by letting $M^i = \{w \in M_n : w_i = x\}$ we have an ordered partition of M, namely, $(M^1, \cdots, M^n)$ and $\exists\, u \in M^i$ such that $t_i(u) \in M^{i+1}$ where $t_i = (x, b_i)$ and b_i is as defined in (i).

(iii) By the *projection theorem* (Theorem 3.2), the definition of M^i and (ii), we have that $\prod_{t_i}^{M^i} \subset M^i$ and $t_i(\prod_{t_i}^{M^i}) \subset M^{i+1}$.

(iv) On the other hand, if $v \in M^{i+1}/t_i(\prod_{t_i}^{M^i})$ then the set of symbols $\{v_l, l < i+1\} = \{b_l, l < i+1\}$ by (i) and by the order imposed on *Ch*.

(v) (iii), (iv), and the fact that $v_{i+1} = x$ allow us to conclude that $M^{i+1} \subseteq \Psi(Ch^i)$

(vi) where Ch^i is the saturated chain of *Ch* between $t_{i-1}(p)$ and $t_i^{-1}(q)$, with the understanding that $t_0(p)$ should be taken as I. By Fact 4.2 (iii) (c) we know that $\Psi(Ch^i)$ = FIRST_HALF $(\Psi(Ch^i))$ x $\{x\}$ x SECOND_HALF $(\Psi(Ch^i))$ where FIRST_HALF $(\Psi(Ch^i)) \subset S_{\{b_l, l<i+1\}}$ and SECOND_HALF $(\Psi(Ch^i)) \subset S_{\Sigma-\{b_l, l<i+1\}}$ are consistent and connected sets, each of which contains a pseudochain. Therefore, |FIRST_HALF $(\Psi(Ch^i))| \leqq |M_i|$ and |SECOND_HALF $(\Psi(Ch^i))| \leqq |M_{n-i-1}|$, which in turn imply by (v) that $|M^{i+1}| \leqq |M_i| * |M_{n-i-1}|$.

(vii) This, together with (ii) above, give us $|M_n| = \sum_{i=0}^{n-1} |M^{i+1}| \leqq \sum_{i=0}^{n-1} |M_i| * |M_{n-i-1}|$ with $|M_0| = 1$, $|M_1| = 1$, $|M_2| = 2$, $|M_3| = 4$.

Inequality (vii) and the fact that the Catalan numbers $\{C_n\}$ satisfy that $C_n = \sum_{i=0}^{n-1} C_i * C_{n-i-1}$ with the same boundary conditions allow us to apply induction on n to get that $|M_n| < C_n$ for every $n > 2$. □

COROLLARY 4.1. *If* M_n *is a maximal consistent subset of* S_Σ *of diameter* $\mathrm{diam}(M_n) = \binom{n}{2}$ *then* $|M_n| < 4^{n-1}$.

Proof. The proof follows from the preceding theorem and from the fact that $C_n \leqq 4^{n-1}$. □

Remarks. The preceding results suggest the possibility of studying the structure of *maximal consistent* sets by looking at them as representing a certain restricted collection of binary trees or as a certain subcollection of stack permutations (de Bruijn [11]). The multiple interpretations offered in the literature to the Catalan numbers, C_n, (de Bruijn [11], Feller [14], Gardner [16], Klamer [18]), could be a good source of ideas to shed new light on the problem in question. This approach has not yet been pursued.

The unexpected relationship between C_n and $|M_n|$ established in Theorem 4.3 offers the (unique) best known upper bound at present. In a forthcoming paper we will prove that $|M_n|$ is not bounded by 2^n for all n, as was conjectured in [2]. We conjecture that in general *any consistent set* $M \subset S_\Sigma$ *satisfies that* $|M| < 4^{|\Sigma|-1}$ *for* $|\Sigma| > 2$ *and that if M contains a maximal pseudochain in the weak Bruhat order then* $|M| < 3^{|\Sigma|-1}$.

We suspect that a general bound for *connected consistent* sets between $3^{|\Sigma|-1}$ and $4^{|\Sigma|-1}$ is a very hard result to obtain because the structure of general connected sets is as random as that of unconnected ones. Moreover, relating *connected consistent* sets to *unconnected* ones appears to be a very hard problem. In Abello [1] we present a very surprising bijection of this type that gives a unified view of several constructions (connected and unconnected) offered in the past.

Conclusions. We have seen that maximal pseudochains in $\langle S_\Sigma, \beta\rangle$ are a very important substructure of those *maximal consistent sets* which contain them. From the *Arrow's Impossibility Theorem* point of view (Abello [4], Arrow [5]), the results obtained here indicate that the majority rule produces transitive results if the collection of voters as a whole (at least in the *extensible* cases covered by Theorem 3.2), can be partitioned into no more than $(n^2 + n)/2$ groups that can be ordered according to the level of disagreement they have with respect to a fixed permutation p. On the other hand, by viewing S_Σ as a *Coxeter* group (Benson and Grove [6], Bourbaki [10], Coxeter and Moser [13], Stanley [23]), these results provide a "*novel*" *interpretation* of the following partition of the collection Ω of maximal chains in the *weak Bruhat order*. Namely, if for Ch and $Ch' \in \Omega$ we let M_{Ch} and $M_{Ch'}$ be the maximal consistent sets containing them, respectively, then the relation $\sim$ given by $Ch \sim Ch'$ if and only if $M_{Ch} = M_{Ch'}$ partition Ω and our results say that $\langle \cup_{Ch' \sim Ch} Ch', \beta\rangle$ is an *upper semimodular sublattice* of $\langle S_\Sigma, \beta\rangle$ such that $\lceil \cup_{Ch' \sim Ch} Ch' \rceil \leqq$ the $|\Sigma|$th Catalan number. Now, if $\gamma = (t_1, \cdots, t_l)$ is a *reduced decomposition of* w_0 = minimum element in $\langle S_\Sigma, \beta\rangle$, any other reduced decomposition of w_0 may be obtained from γ by using two types of transformations known as Coxeter relations of type I and of type II (see Benson and Grove [6]). Our Projection Theorem (Theorem 3.2) shows that $Ch \sim Ch'$ if and only if Ch' may be obtained from Ch by using transformations of type I only; therefore, we have obtained a "new" combinatorial interpretation of the collection of chains which can be obtained from one another by using Coxeter transformations of type I or type II exclusively. Namely, for $Ch' \in \Omega$, if $\Omega_{Ch'} = \{Ch \in \Omega\colon Ch$ can be obtained from Ch' by using Coxeter transformations of type I only$\}$ then the set $\cup_{Ch \in \Omega_{Ch'}} Ch$ does not contain a cyclic triple (or Latin square) in the sense of Definition 1.1.

If one is puzzled by the fact that we never said what these transformations were, it should suffice to say that what we call transformations of type I correspond to interchanging t_i and t_{i+1}, in the reduced decomposition γ of w_0, if and only if they are "disjoint."

We close with the following question: What is the corresponding combinatorial interpretation of the projection theorem for general coxeter groups?

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